Math 246B Lecture 6 Notes

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1 Subharmonicity and Convexity

1.1 Jensen's inequality and composition of convex functions with subharmonic functions

Last time, we showed that $u \in C^2(\Omega)$ is subharmonic iff $\Delta u \geq 0$ in Ω.

Remark 1.1. Let $u \in SH(\Omega)$ be such that $u \not\equiv$ on any component (so $u \in L^!_{\text{loc}}$). Approximating u by a decreasing sequence of smooth, subharmonic functions, one may show that $\int u \Delta \varphi \, dx \geq 0$ for all $0 \leq \varphi \in C^2(\Omega)$ such that $\varphi = 0$ outside a compact subset of Ω.

Theorem 1.1. *Let Ω be open, $u \in SH(\Omega)$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be increasing and convex. Then $\varphi \circ u \in SH(\Omega)$ (we define $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$).*

Example 1.1. If $f \in \operatorname{Hol}(\Omega)$, then $|f|^a \in SL(\Omega)$ for any $a > 0$. Write $u = \log |f|$ and $\varphi(t) = e^{at}$, where $a > 0$.

To proof this theorem, we need the following general inequality for convex functions.

Proposition 1.1 (Jensen's inequality). *Let $I \subseteq \mathbb{R}$ be an open interval, and let $\psi : T \to \mathbb{R}$ be convex. Let (Ω, μ) be a measure space equipped with a probability measure ($\mu(\Omega) = 1$). Let $f \in L^1(\Omega, I)$. Then*

$$\psi\left(\int f \, d\mu \right) \leq \int \psi_0 f \, d\mu.$$

Proof. Let $I = (a, b)$, and let $c = \int f \, d\mu \in (a, b)$. If for $a < t_1 < c < t_2 < b$, $c = \alpha t_1 = (1 - \alpha) t_2$, where $\alpha = (t_2 - c)/(t_2 - t_1)$, then $\psi(c) \leq \alpha \psi(t_1) + (1 - \alpha) \psi(t_2)$. After some algebra, we get

$$\frac{\psi(c) - \psi(t_1)}{c - t_1} \leq \frac{\psi(t_2) - \psi(c)}{t_2 - c}.$$

So

$$\underbrace{\sup_{t_1 < c} \frac{\psi(c) - \psi(t_1)}{c - t_1}}_{= \psi'_{\text{left}}(c)} \leq \underbrace{\inf_{t_2 > c} \frac{\psi(t_2) - \psi(c)}{t_2 - c}}_{= \psi'_{\text{right}}(c)},$$

where these are the left and right derivatives of φ at c. Then $\psi(t) \geq \psi(c) + \psi_{\text{right}}(c)(t-c)$ for all $t \in I$. That is the tangent line at c lies below the graph of ψ. It follows that

$$\int \psi(f)\, d\mu \geq \psi\left(\int f\, d\mu\right) + \psi'_{\text{right}}(c)\left(\cancelto{0}{\int f - c}\right). \qquad \square$$

Now let's prove the theorem.

Proof. Let $\{|x-a| \leq R\} \subseteq \Omega$. Then

$$u(a) \leq \frac{1}{2\imath R} \int_{|y|=R} u(a+y))\, ds(y).$$

Applying Jensen's inequality,

$$\varphi(u(a)) \leq \frac{1}{2\pi i} \int_{|y|=R} \varphi(u(a+y))\, ds(y).$$

We also check that $\varphi \circ u$ is upper semicontinuous (since φ is continuous). We get that $\varphi \circ u \in SH(\Omega)$. $\square$

1.2 Maximality bounds in an annulus

Theorem 1.2. *Let u be subharmonic in $0 \leq R_1 < |x| < R_2 \leq \infty$, and let $M(r) = \max_{|x|=r} u(r)$. Then $M(r)$ is a convex function of $\log(r) \in (\log(R_1), \log(R_2))$: if $r_1, r_2 \in (R_1, R_2)$ and $0 \leq \lambda \leq 1$, then*

$$M(r_1^{\lambda} r_2^{1-\lambda}) \leq \lambda M(r_1) + (1-\lambda) M(r_2).$$

If u is subharmonic in $|x| < R$, then $M(r)$ is an increasing function of r.

Proof. We claim that if I is an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ is convex if any only if for any compact interval $J \subseteq I$ and any linear function L,

$$\sup_J (f - L) = \sup_{\partial J} (f - L).$$

This follows from the fact that the graph of f on J lies beneath the chord connecting the endpoints.

Using this characterization of convexity, we have to show that if $a, b \in \mathbb{R}$ are such that $\tilde{M}(r) = M(r) = a \log(r) - b$ is such that $M(r_j) \leq 0$ for $j = 1, 2$, then $\tilde{M}(r) \leq 0$ when $r_1 \leq r \leq r_2$. If we set $v(x) = u(x) - a \log|x| - b$, then $v(x) \in SH(R_1 < |x| < R_2)$ since $a \log|x| - b$ is harmonic. Then $\tilde{M}(r) = \max_{|x|=r} v(x)$. If $v(x) \leq 0$ when $|x| = r_1$ and $|x| = r_2$, then $v(x) \leq 0$ for $r_1 \leq |x| \leq r_2$ by the maximum principle. Therefore, $\tilde{M}(r) \leq 0$ for $r_1 \leq r \leq r_2$. This shows that $M(r)$ is convex as a function of $\log(r)$.

If $u \in SH(|x| < R)$, then $M(r)$ increases by the maximum principle applied to u. $\square$

Corollary 1.1 (Hadamard's three circle theorem)**.** *Let* $f \in \mathrm{Hol}(R_1 < |z| < r_2)$*, and let* $M(r) = \max_{|z|=r} |f(z)|$*. Then* $\log(M(r))$ *is a convex function of* $\log(r)$*: if* $r_1, r_2 \in (R_1, R_2)$ *and* $0 \le \lambda \le 1$*, then*

$$M(r_1^\lambda r_2^{1-\lambda}) \le M(r_1)^\lambda M(r_2)^{1-\lambda}.$$

Proof. Apply the theorem to $u = \log|f|$. □

Remark 1.2. This inequality is much sharper than what we get from the usual maximum principle applied to $|f|$: $M(r_1^\lambda r_2^{1-\lambda}) \le \max(M(r_1), M(r_2))$.

Next time, we will prove the following result (and more).

Proposition 1.2. *If* $u \in SH(|x| < R)$*, then the average*

$$I(r) := \frac{1}{2\pi r} \int_{|y|=r} u(y)\, ds(y).$$

is a convex function of $\log(r)$ *which is increasing.*